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2004 J. Phys. A: Math. Gen. 37 3869

(http://iopscience.iop.org/0305-4470/37/12/010)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 3869–3879 PII: S0305-4470(04)72443-9

Husimi's $Q(\alpha)$ function and quantum interference in phase space

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Received 21 November 2003, in final form 3 February 2004

Published 10 March 2004

Online at stacks.iop.org/JPhysA/37/3869 (DOI: 10.1088/0305-4470/37/12/010)

Abstract

We discuss a phase-space description of the photon number distribution of non-classical states which is based on Husimi's $Q(\alpha)$ function and does not rely on the WKB approximation. We illustrate this approach using the examples of displaced number states and two photon coherent states and show it to provide an efficient method for computing and interpreting the photon number distribution. This result is interesting in particular for the two photon coherent states which, for high squeezing, have the probabilities of even and odd photon numbers oscillating independently.

PACS numbers: 42.50.Dv, 42.50.Ar

1. Introduction

The oscillations in the photon number distribution are an interesting feature common to various kinds of light states, which may be taken as a signal of non-classical behaviour of those states. They were first computed for squeezed states [1, 2] and interpreted as an interference effect in phase space [1–4]. They have also been the subject of experimental investigations [5] in connection with the properties of Wigner's distribution. Since in the actual experimental situation, many factors such as the detector properties may modify the conclusions of the theoretical analysis; it is of interest to understand better the physics involved and to develop our intuition of how the oscillations may appear and eventually disappear.

For our analysis, we consider a single mode of the electromagnetic field. For this case, the picture of quantum interference of states in phase space was developed as a generalization of the Bohr–Sommerfeld description of quantum states as finite areas in phase space [1–4, 6]. The total area of each state is fixed by the normalization requirement $\langle\psi|\psi\rangle = 1$. The area associated with a Fock or number state $|m\rangle$ is a circular ring centred at the origin with interior radius $r_m^- = \sqrt{2m}$ and exterior radius $r_m^+ = \sqrt{2m+2}$. The areas associated with coherent states and squeezed states are obtained by displacing, or squeezing and displacing the one associated with the vacuum state. The inner product $\langle\psi|\phi\rangle$ between two quantum states is then related with the overlapping area of the states in the phase space. The intersecting region may have in some interesting cases more than one component. Since the probability amplitude $\langle m|\psi\rangle$ is a complex number and the overlapping areas are real, it results naturally to associate with each of these components a phase in order to reproduce the quantum mechanical results. The probability amplitude then acquires the following structure:

$$\langle\chi|\psi\rangle = \sum_i \sqrt{A^i_{\chi\psi}(\psi)}\exp\left(\phi^i_{\chi\psi}(\psi)\right) \tag{1}$$

where $A^i_{\chi\psi}(\psi)$ and $\phi^i_{\chi\psi}(\psi)$ are the ith component of the overlapping area and their assigned phase, respectively.

The presence of the areas $A^i_{\chi\psi}(\psi)$ in this expression is very natural and physically appealing although the fact that what appears is the square root of the areas does not allow any direct geometrical method to derive equation (1). On the other hand, the values that have to be chosen for the phases $\phi^i_{\chi\psi}(\psi)$ are not evident from the geometry of the phase space. Dowling *et al* [4] worked out a quite general equivalent of equation (1) for the probability amplitude of the eigenstates of two different Hamiltonian operators using the WKB approximation. They obtained an explicit expression of the phases where it is possible to recognize a geometrical content. Oscillations in the photon statistics for displaced states, two photon squeezed states and for squeezed number states may also be studied with this methodology but this approach is limited by the validity of the WKB approximation [6, 7].

Some time later, Milburn [8] showed that the interference effects in phase space may also be understood by considering the properties of Husimi's function

$$Q(\alpha) = \frac{1}{\pi}|\langle\alpha|\psi\rangle|^2$$

for the state under study. This author notes that the over-completeness of the coherent states allows us to rewrite the probability amplitude $\langle m|\psi\rangle$ as a phase-space integral,

$$\langle m|\psi\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\,\langle m|\alpha\rangle\langle\alpha|\psi\rangle. \tag{2}$$

The functions $\langle m|\alpha\rangle$ and $\langle\alpha|\psi\rangle$ could be interpreted as the phase-space probability amplitudes for the states $|m\rangle$ and $|\psi\rangle$, respectively. Each of these functions is proportional to the function $Q(\alpha)$ of the corresponding state. Then,

$$\langle m|\alpha\rangle = \sqrt{\pi\,Q_m(\alpha)}\,\mathrm{e}^{\mathrm{i}\phi_m(\alpha)} \tag{3}$$

$$\langle\alpha|\psi\rangle = \sqrt{\pi\,Q_\psi(\alpha)}\,\mathrm{e}^{\mathrm{i}\phi_\psi(\alpha)}. \tag{4}$$

One may then approximate the integral in (2), by restricting the domain of integration to the phase-space region where the product of the two probability amplitudes is appreciably different from zero and by identifying the regions where the Q function concentrates with the phase-space regions assigned to the states, this approximation is equivalent to consider the integration domain as the overlapping areas between the different states. Following this line of thought, in this paper we discuss in detail the photon number distributions for the displaced number states and the two photon coherent states. We show that this approach allows us to identify the areas and phases for the phase-space description without rendering to the WKB approximation. This result is interesting in particular for the two photon coherent states which have, for high squeezing, probabilities of even and odd photon numbers oscillating independently.

2. Photon statistics for displaced number states

Let us first consider the example of the displaced number states. They are defined through the action of the displacement operator $D(\beta) = \exp(\beta a^\dagger - \beta^* a)$ on the Fock states $|n\rangle$,

$$|n, \beta\rangle = D(\beta)|n\rangle. \tag{5}$$

For simplicity we work with real β.

We compute the photon distribution using the method of generating functions. Consider a coherent state $|\alpha\rangle = D(\alpha)|0\rangle$ with real α. We have,

$$D(\beta)|\alpha\rangle = |\alpha + \beta\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} D(\beta)|n\rangle. \tag{6}$$

Then,

$$\langle m|D(\beta)|n\rangle = \frac{1}{\sqrt{n!}} \left\{ \left(\frac{\partial}{\partial\alpha}\right)^n \left(\mathrm{e}^{|\alpha|^2/2}\langle m|\alpha + \beta\rangle\right)\right\}_{\alpha=0} \tag{7}$$

with

$$\langle m|\alpha + \beta\rangle = \frac{1}{\sqrt{m!}} \exp\left(-\frac{|\alpha + \beta|^2}{2}\right)(\alpha + \beta)^m. \tag{8}$$

As in other cases the photon number distribution $P_{mn}(\beta) = |\langle m|n, \beta\rangle|^2$ is oscillating (see figure 6).

To develop the phase-space description, consider the phase-space amplitudes,

$$\begin{aligned}\langle m|\alpha\rangle &= \frac{1}{\sqrt{m!}} \mathrm{e}^{-|\alpha|/2}\alpha^m \\ &= |\langle m|\alpha\rangle|\, \mathrm{e}^{\mathrm{i}\phi_m(\alpha)}\end{aligned} \tag{9}$$

and

$$\begin{aligned}\langle\alpha|n, \beta\rangle &= \frac{1}{\sqrt{n!}} \exp\left(\frac{\beta\alpha^* - \alpha\beta^*}{2}\right) \exp\left(-\frac{|\alpha - \beta|^2}{2}\right)(\alpha^* - \beta^*)^n \\ &= |\langle\alpha|n, \beta\rangle|\, \mathrm{e}^{-\mathrm{i}\phi_{n,\beta}(\alpha)}.\end{aligned} \tag{10}$$

The phases here are better expressed in terms of the real and imaginary parts of $\alpha = x + \mathrm{i}y$ and take the form

$$\phi_m(\alpha) = m \arctan\left(\frac{y}{x}\right) \tag{11}$$

and

$$\phi_{n,\beta}(\alpha) = \frac{\alpha\beta^* - \beta\alpha^*}{2\mathrm{i}} + n \arctan\left(\frac{y}{x - \beta}\right). \tag{12}$$

In figure 1 we show the phase-space probability amplitude module $|\langle m|\alpha\rangle|$ for a number state with $m = 3$. This function recalls the Bohr–Sommerfeld ring associated with the corresponding number state except for the fact that here the mean radius is $|\alpha| = \sqrt{m}$. This suggests to associating the number state $|m\rangle$ with the circular ring centred at origin and located between the radii $\sqrt{m - 1/2}$ and $\sqrt{m + 1/2}$. Note also that the phase-space amplitude $|\langle\alpha|n, \beta\rangle|$ for the displaced number state $|n, \beta\rangle$ is obtained by displacing the amplitude $|\langle\alpha|n\rangle|$.

In figure 2 we show the product $|\langle m|\alpha\rangle\langle\alpha|n, \beta\rangle|$ for $m = 100, n = 3$ and $\beta = 10.1$. Observe that there are two overlapping regions where both terms are appreciably not vanishing. But also note that the intersection areas with this prescription are not in the same position

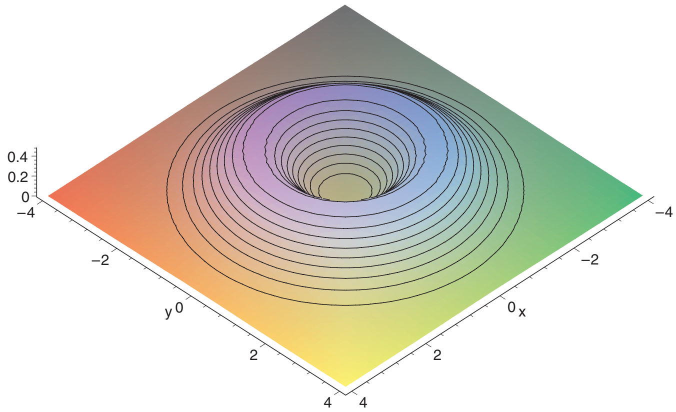

Figure 1. $|\langle m|\alpha\rangle|$ for the Fock state with $m = 3$.

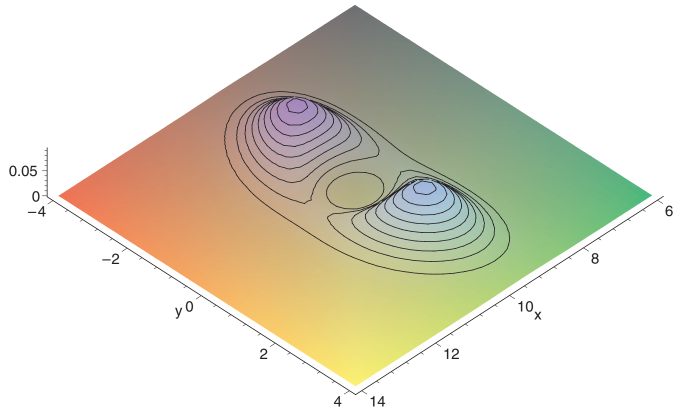

Figure 2. $|\langle m|\alpha\rangle\langle\alpha|n, \beta\rangle|$ for the Fock state with $m = 100$ and a displaced state with $n = 3$ and $\beta = 10.1$.

where they would appear if working with the Bohr–Sommerfeld bands. Now they may be localized at the intersection points of a circumference of radius $\sqrt{m}$ centred at the origin and a second circumference of radius $\sqrt{n}$ displaced by β, as shown in figure 3. The intersection points are then given by $\alpha_+ = x_0 + \mathrm{i}y_0$ and $\alpha_- = x_0 - \mathrm{i}y_0$, with

$$x_0 = \frac{\beta^2 + m - n}{2\beta} \tag{13}$$

and

$$y_0 = \sqrt{m - x_0^2}. \tag{14}$$

Now we approximate the phase-space integral of equation (2) for the photon number probability amplitude $\langle m|n, \beta\rangle$ by taking instead of (9) and (10), uniform contributions on the intersection of the rings described above with the angles (11) and (12) evaluated at the points given by (13) and (14). The probability amplitude has then the structure,

$$\langle m|n, \beta\rangle = \frac{\sqrt{A_{mn}}}{\pi}\,\mathrm{e}^{\mathrm{i}\psi} + \frac{\sqrt{A_{mn}}}{\pi}\exp\left(-\mathrm{i}\psi_{n,\beta}^{(m)}\right) \tag{15}$$

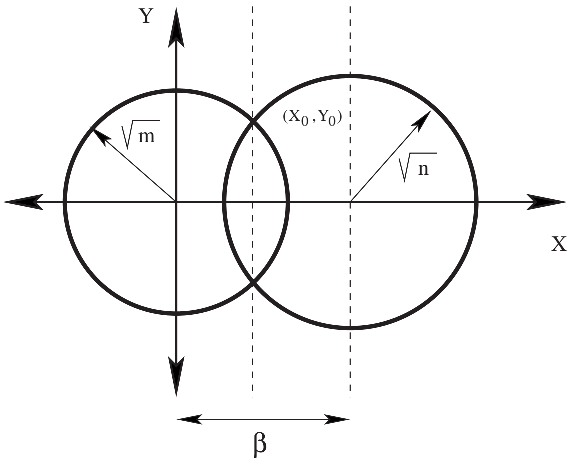

Figure 3. Intersection points in phase space.

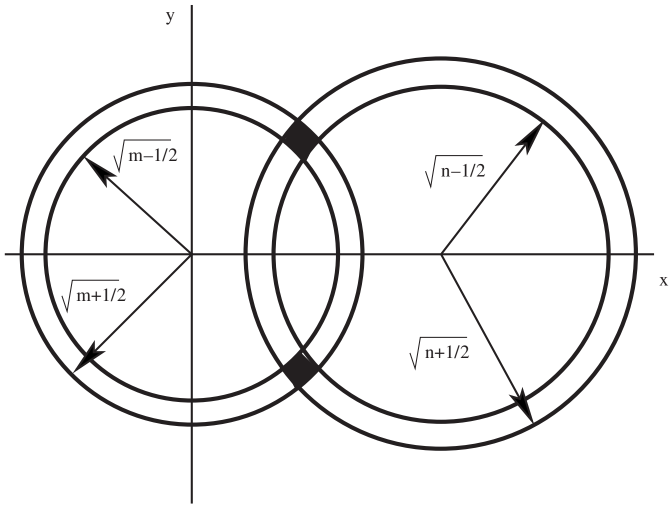

Figure 4. Overlapping areas between the Fock state $|m\rangle$ and the displaced number state $|n, \beta\rangle$.

where

$$\begin{aligned} \psi_{n,\beta}^{(m)} &= \phi_m(\alpha_+) - \phi_{n,\beta}(\alpha_+) \\ &= m \arctan\left(\frac{y_0}{x_0}\right) - n \arctan\left(\frac{y_0}{x_0 - \beta}\right) - \beta y_0 \end{aligned} \tag{16}$$

and A_{mn} are the overlapping areas between the rings corresponding to the number and displaced number states as shown in figure 4.

The computation of these areas is analogous to the one presented in [3, 4] for the Bohr–Sommerfeld strips. They are given by

$$\begin{aligned} A_{mn} = \tfrac{1}{2}\Bigg(& a\left(\sqrt{m+\tfrac{1}{2}}, \sqrt{n+\tfrac{1}{2}}\right) - a\left(\sqrt{m-\tfrac{1}{2}}, \sqrt{n+\tfrac{1}{2}}\right) \\ & - a\left(\sqrt{m+\tfrac{1}{2}}, \sqrt{n-\tfrac{1}{2}}\right) + a\left(\sqrt{m-\tfrac{1}{2}}, \sqrt{n-\tfrac{1}{2}}\right)\Bigg) \end{aligned} \tag{17}$$

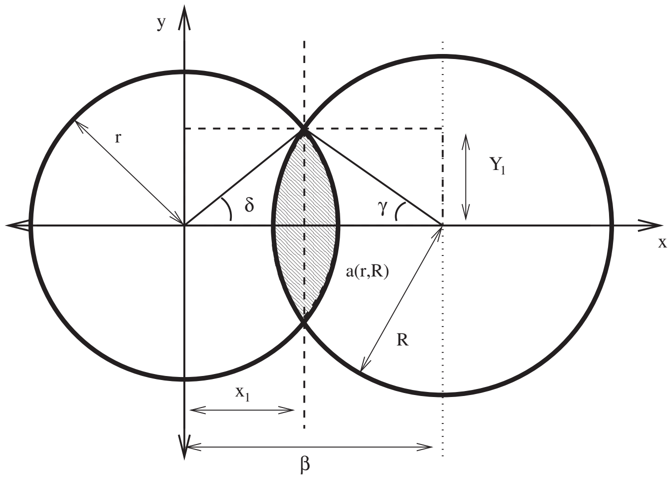

Figure 5. Internal areas used to define A_{mn}.

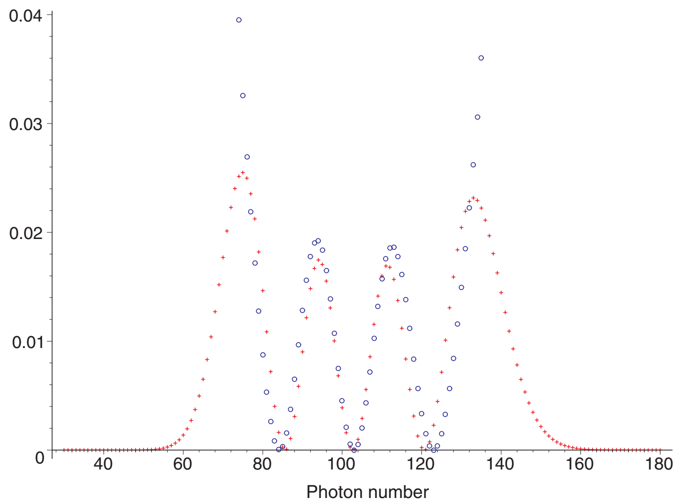

Figure 6. Photon statistics P_{mn} for a displaced number state $|n, \beta\rangle$ with $n = 3$ and $\beta = 10.1$. Exact computation: crosses. Approximation: circles.

where $a(r, R) = r^2\delta + R^2\gamma - \beta Y_1$ is the internal area between the two circular paths with radii r and R as shown in figure 5, with $\left(X_1 = \frac{r^2+\beta^2-R^2}{2\beta}, Y_1 = \sqrt{r^2 - X_1^2}\right)$ the intersection point of the circumferences, $\delta = \arctan\left\{\frac{Y_1}{X_1}\right\}$ and $\gamma = \arctan\left\{\frac{Y_1}{\beta - X_1}\right\}$.

Figure 6 shows the exact and approximate results of P_{mn} for a displaced state with $n = 3$ and $\beta = 10.1$ showing a good qualitative and quantitative behaviour for the values for which the assigned rings actually overlap.

The comparison of the result in equation (16) which has the direct geometrical interpretation

$$\psi_{n,\beta}^{(m)} = a(\sqrt{m}, \sqrt{n}) - n\pi \tag{18}$$

with the one obtained by Dowling *et al* using the WKB approach is shown in figure 7.

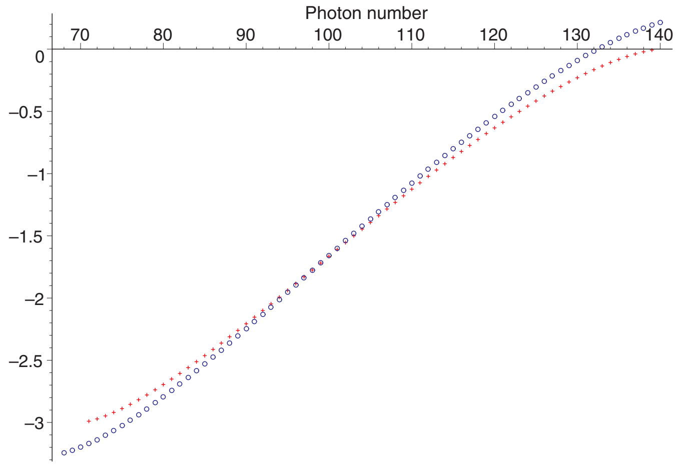

Figure 7. A comparison between the phases. Crosses: $(\psi + n\pi)/\pi$. Circles: $(\psi^{\mathrm{WKB}} + n\pi)/\pi$.

The WKB phase is given by [4, 9]

$$\psi^{\mathrm{WKB}} = -\left(m+\frac{1}{2}\right)\arcsin\left[\frac{x_c(m,n)}{\sqrt{2m+1}}\right] + \left(n+\frac{1}{2}\right)\arcsin\left[\frac{x_c(m,n)-\sqrt{2}\beta}{\sqrt{2n+1}}\right] \\ - \frac{\sqrt{2}\beta}{2}\sqrt{2m+1-x_c(m,n)^2} - \frac{(n-m)\pi}{2} + \frac{\pi}{4} \tag{19}$$

where

$$x_c(m,n) = \frac{m-n}{\sqrt{2}\beta} + \frac{\sqrt{2}\beta}{2}. \tag{20}$$

The phases differ sensibly for zero and high photon number where the WKB approximation fails to reproduce the distribution [7].

3. Two photon coherent states

As a second illustration let us consider the two photon coherent states. These are obtained by the application of the squeezing operator $S(r) = \exp\left\{\frac{r}{2}a^2 - \frac{r}{2}(a^\dagger)^2\right\}$ on the coherent state $|\alpha\rangle$. That is

$$|\beta, r\rangle = S(r)|\beta\rangle. \tag{21}$$

Again we take β and ξ real. The photon statistics in this case is given by [10, 11]

$$P_n = |\langle n|\beta, r\rangle|^2 = \frac{(\tanh(r))^n}{2^n n! \cosh(r)} \exp\{\beta^2(\tanh(r)-1)\} \left| H_n\left(\frac{\beta}{\sqrt{2\cosh(r)\sinh(r)}}\right)\right|^2 \tag{22}$$

with $H_n(x)$ being the order n Hermite polynomial.

For $r \ll 1$ the photon statistics resembles that of a coherent state (although in fact the statistics is sub-Poissonian). As the squeezing increases, oscillations appear in the distribution which are characteristic of the phase-space interference. But then for higher values of r (see figure 8) the distribution develops different oscillating behaviour for odd and even photon number. This effect, which, as we show below, may be understood in terms of phase-space

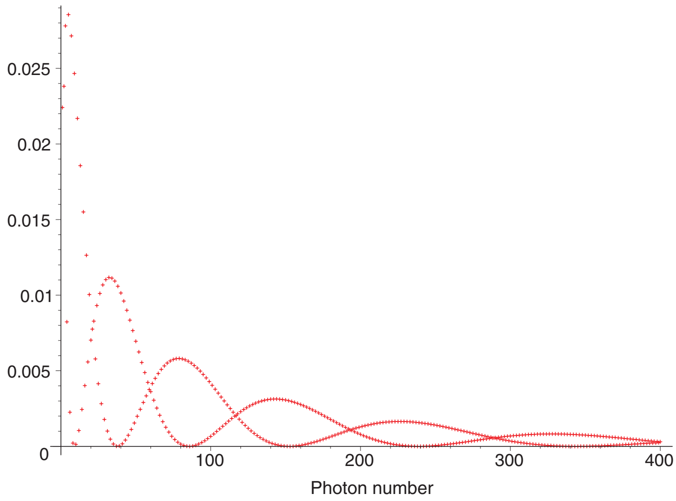

Figure 8. Photon statistics for a two photon coherent state with $\beta = 5.1$ and $r = 3$.

interference. Following the same line as in the last section take equations (9) and (11) and consider

$$\begin{aligned}\langle\alpha|\beta, r\rangle &= \sqrt{\operatorname{sech}(r)}\exp\left\{-\tfrac{1}{2}(|\alpha|^2+\beta^2)+\alpha^*\beta\operatorname{sech}(r)-\tfrac{1}{2}((\alpha^*)^2-\beta^2)\tanh(r)\right\}\\ &= |\langle\alpha|\beta, r\rangle|\,\mathrm{e}^{\mathrm{i}\phi_{\beta,r}(\alpha)}\end{aligned} \tag{23}$$

which define

$$\phi_{\beta,r}(\alpha) = -y\beta\operatorname{sech}(r) + xy\tanh(r). \tag{24}$$

It is not difficult to show that the phase-space probability amplitude $|\langle\alpha|\beta, r\rangle|$ of the two photon coherent states concentrates on an ellipse centred at $\beta\exp\{-r\}$, with one semi-axis of magnitude e^{-r} oriented through the X-axis and the other semi-axis of magnitude e^{r} oriented through the Y-axis. We represent the state by the internal region of the ellipse,

$$(x-\beta\,\mathrm{e}^{-r})^2\,\mathrm{e}^{2r} + y^2\,\mathrm{e}^{-2r} = 1 \tag{25}$$

of area π. For high values of r the ellipse approximates to a vertical line (see figure 9). In this limit the intersection points with the circumference of radius $\sqrt{m}$ (which we are using to represent the number states) are $\alpha_+ = X_2 + \mathrm{i}Y_2$ and $\alpha_- = X_2 - \mathrm{i}Y_2$, with

$$X_2 = \beta\exp\{-2r\} \qquad Y_2 = \sqrt{m - X_2^2}. \tag{26}$$

As in the previous case, we may write the probability amplitude $\langle m|\beta, r\rangle$ as

$$\langle m|\beta, r\rangle = \frac{\sqrt{A_m}}{\pi}\,\mathrm{e}^{\mathrm{i}\psi_{\beta,r}} + \frac{\sqrt{A_m}}{\pi}\,\mathrm{e}^{-\mathrm{i}\psi_{\beta,r}^{(m)}} \tag{27}$$

where the phases are evaluated at the intersection points are given by

$$\begin{aligned}\psi_{\beta,r}^{(m)} &= \phi_m(\alpha_+) + \phi_{\beta,r}(\alpha_+)\\ &= m\arctan\left\{\frac{Y_2}{X_2}\right\} - Y_2\beta\operatorname{sech}(r) + X_2Y_2\tanh(r)\end{aligned} \tag{28}$$

and A_m is one of the shadowed areas in figure 10. It has the value

$$A_m = \Delta y\Delta x \tag{29}$$

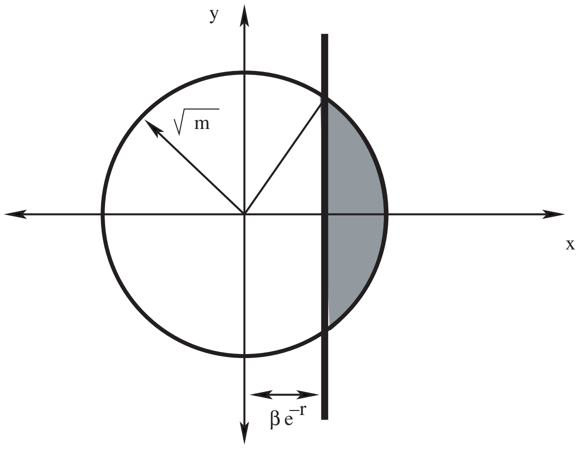

Figure 9. Intersecting points and the phase area for $\langle m|\beta, r\rangle$.

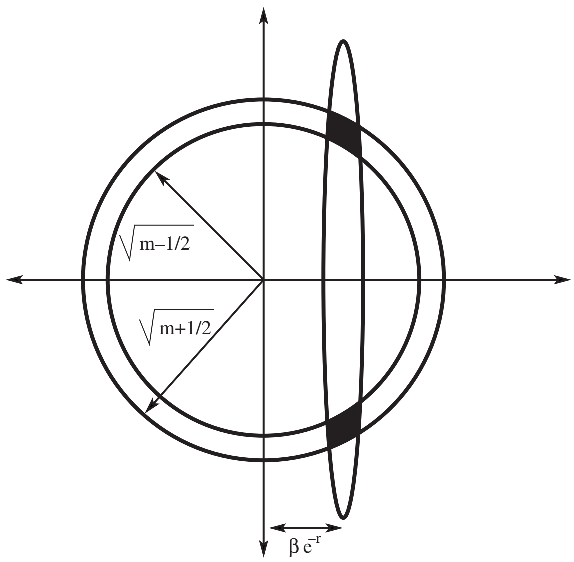

Figure 10. Overlapping areas for $\langle m|\beta, r\rangle$.

with

$$\Delta y = \sqrt{m + \tfrac{1}{2} - \beta^2\,\mathrm{e}^{-2r}} - \sqrt{m - \tfrac{1}{2} - \beta^2\,\mathrm{e}^{-2r}}$$
$$\Delta x = 2\,\mathrm{e}^{-r}\sqrt{1 - (m - \beta^2\,\mathrm{e}^{-2r}))\,\mathrm{e}^{-2r}}$$

Figures 11 and 12 show the comparison between the exact and approximate results for the photon statistics of a two photon coherent state.

To get a geometrical interpretation of the two branches in the photon distribution consider again the phase evaluated at (28). It may be rewritten in the form,

$$\psi_{\beta,r} = m\phi - X_2Y_2\left(\frac{\mathrm{e}^r}{\cosh(r)} - \frac{\sinh(r)}{\cosh(r)}\right) = m\phi - X_2Y_2. \tag{30}$$

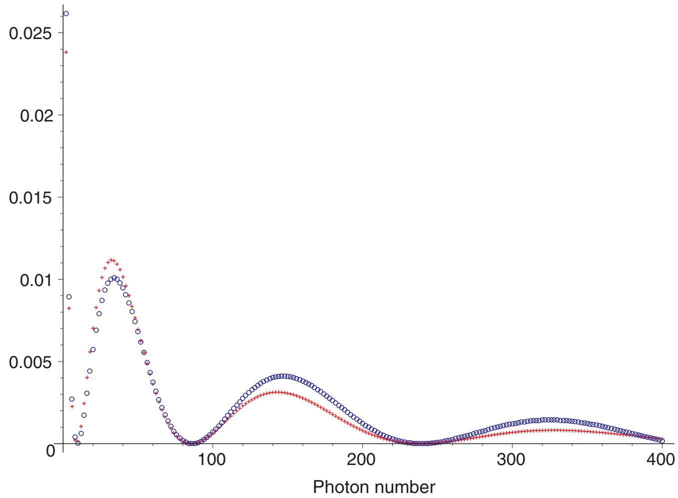

Figure 11. P_m with m even for a two photon coherent state with $\beta = 5.1$ and $r = 3$. Circles: exact result. Crosses: approximate result.

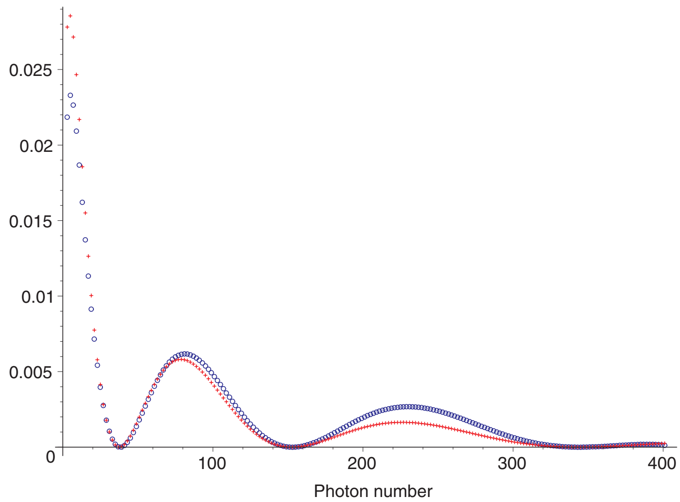

Figure 12. P_m with m odd for a two photon coherent state with $\beta = 5.1$ and $r = 3$. Circles: exact result. Crosses: approximate result.

From this expression follows that the phase $\psi_{\beta,r}$ is represented by the shadowed area in figure 9. For high values of r, this phase may be written as

$$\psi_{\beta,r} = m\pi/2 - 2X_2Y_2. \tag{31}$$

In this case the probability is given by

$$P_m = \frac{A_m}{\pi}\cos^2(\psi_{\beta,r}) = \frac{A_m}{2\pi}[1 + (-1)^m \cos(4X_2Y_2)] \tag{32}$$

which explains the differences in the probabilities of odd and even photon numbers.

4. Conclusion

In the approach to quantum phase-space interference presented in this paper, we show a general set-up, based on the properties of Husimi's $Q(\alpha)$ function, for the representation of quantum states as regions in phase space with a prescription for assigning phases to the different overlapping regions which has a geometrical input. We illustrate the method discussing the photon number distribution of the displaced number states and of the two photon coherent states. For the latter at high squeezing, we show that the distribution develops different oscillating behaviour for odd and even photon numbers. This effect (also displayed by squeezed states) is understood in terms of phase-space interference and maybe, in principle, suitable for experimental verification.

Acknowledgments

JS thanks the Abdus Salam International Centre for Theoretical Physics for a regular associate fellowship. This work was supported by Did-Usb grant Gid-30 and by Fonacit grant G-2001000712.

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